

Infinitary Logic Formalization Blueprint

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Chapter 1

Infinitary Logic

This project formalizes results in infinitary logic in Lean 4 using Mathlib. The primary focus is the logic $\mathcal{L}_{\omega_1\omega}$, which extends first-order logic with countable conjunctions and disjunctions, though the formalization treats the more general $L_{\infty\omega}$ (allowing arbitrary conjunctions and disjunctions) where possible.

1.1 Main Results

The formalization establishes five headline theorems:

- **Scott characterization.** Every countable structure is characterized up to isomorphism by a sentence of $\mathcal{L}_{\omega_1\omega}$ (Theorem 9).
- **Scott rank below ω_1 .** The Scott rank of any countable structure is a countable ordinal (Theorem 11).
- **Karp's theorem.** Two structures are $L_{\infty\omega}$ -equivalent if and only if there exists a back-and-forth system between them (Theorem 16).
- **Model existence.** Every consistent set of sentences of $\mathcal{L}_{\omega_1\omega}$ has a model (Theorem 17).
- **Countable refinement hypothesis.** The back-and-forth hierarchy on any countable structure stabilizes in countably many steps (Theorem 8).

1.2 Back-and-Forth Equivalence

Back-and-forth equivalence at ordinal levels, following Karp [Kar65].

Definition 1 (Back-and-forth equivalence). Back-and-forth equivalence $\sim_\alpha(a, b)$ for tuples $a \in M^n$, $b \in N^n$ at ordinal α , defined by transfinite recursion: atomic agreement at 0, forth-and-back extension at successor, conjunction at limit.

1.3 Scott Formulas and Sentences

Scott formulas and their characteristic property, following Marker [Mar16, Ch. 2].

Definition 2 (Scott formula). The Scott formula $\sigma_\alpha(a)$ for a tuple $a \in M^n$ and ordinal $\alpha < \omega_1$: an $\mathcal{L}_{\omega_1\omega}$ -formula mirroring the recursive definition of \sim_α .

Theorem 3 (Scott formula characterization). *For countable M and $\alpha < \omega_1$: $\sigma_\alpha(a)$ is realized by b in N if and only if $\sim_\alpha(a, b)$.*

Proof. By ordinal induction using `limitRecOn`: the zero case reduces to atomic diagrams, the successor case uses the forth/back quantifier structure of the Scott formula, and the limit case uses countable infimum. \square

1.4 Self-Stabilization

Theorem 4 (Self-stabilization). *For any countable L -structure M , there exists $\alpha < \omega_1$ such that M self-stabilizes completely at α : for every n and tuples $a, a' \in M^n$, $\sim_\alpha(a, a') \iff \sim_{\alpha+1}(a, a')$.*

Proof. For each triple (n, a, a') , the first failure ordinal is 0 or a successor. Take the supremum over countably many triples; $\sup + 1 < \omega_1$. \square

1.5 Scott Sentence

Definition 5 (Stabilization ordinal). The stabilization ordinal of a countable structure M : the least ordinal α such that \sim_α on 0-tuples characterizes isomorphism among countable structures.

Definition 6 (Scott sentence). The Scott sentence of M : the Scott formula for the empty tuple at the stabilization ordinal, $\sigma_{\alpha_0}(\langle \rangle)$.

Theorem 7 (Scott characterization (conditional)). *Assuming the countable refinement hypothesis: the Scott sentence of a countable structure M characterizes M up to isomorphism among countable structures, i.e., $N \models \sigma_M \iff M \cong N$.*

Proof. By the Scott formula characterization at the stabilization ordinal, plus the CRH-conditional stabilization result. \square

Chapter 2

Countable Refinement

The countable refinement hypothesis and its consequences for Scott rank.

2.1 Countable Refinement Hypothesis

Theorem 8 (Countable refinement hypothesis). *For any countable relational language L , countable L -structure M , tuple size n , and tuple $a \in M^n$, the set of refinement ordinals $\{\varepsilon < \omega_1 \mid \exists(N, b) \sim_\varepsilon(a, b) \wedge \neg \sim_{\varepsilon+1}(a, b)\}$ is countable.*

Proof. From self-stabilization: witnesses at α_0 persist at all $\alpha_0 + k$. The chain $\alpha_0, \alpha_0 + 1, \dots$ has limit $\gamma < \omega_1$ which is itself a limit ordinal, giving BFEquiv at γ and hence at all ordinals. \square

Note: Despite its name, the “Countable Refinement Hypothesis” is fully proved in this formalization. The name is historical: it was originally an open conjecture about the countability of refinement ordinals. The proof here proceeds via self-stabilization and transfinite induction.

This countability result drives the downstream stabilization argument: since only countably many ordinals can witness a strict refinement, the hierarchy must stabilize below ω_1 .

Theorem 9 (Scott characterization). *The Scott sentence of a countable structure M characterizes M up to isomorphism among countable structures (unconditional corollary of CRH).*

Proof. Instantiate the conditional characterization theorem with the proved Countable Refinement Hypothesis. \square

2.2 Scott Rank and Height

Definition 10 (Scott rank). The Scott rank of a countable structure M : $\text{sr}(M) = \sup_{m \in M} (\text{er}(m) + 1)$, where $\text{er}(m)$ is the element rank.

Theorem 11 (Scott rank below ω_1). *For any countable L -structure M , $\text{sr}(M) < \omega_1$.*

Proof. By the Countable Refinement Hypothesis, each element rank is countable, and the Scott rank is their supremum. \square

Definition 12 (Scott height). The Scott height of a countable structure M : the least ordinal α such that $\sim_\alpha(a, b) \Rightarrow \sim_{\alpha+1}(a, b)$ for all tuples a, b in any countable structures.

Theorem 13 (Scott height below ω_1). *For any countable L -structure M , $\text{sh}(M) < \omega_1$.*

Proof. By the Countable Refinement Hypothesis, the BF-equivalence hierarchy stabilizes at a countable ordinal for each tuple size, and the Scott height is their supremum. \square

Chapter 3

Karp's Theorem

Karp's theorem characterizes $L_{\infty\omega}$ -equivalence via back-and-forth systems, following Karp [Kar65].

Definition 14 (Potential isomorphism). A potential isomorphism between L -structures M and N : a nonempty family of finite partial maps (pairs of same-length tuples) preserving atomic type and closed under extension in both directions.

Definition 15 ($L_{\infty\omega}$ -equivalence). $L_{\infty\omega}$ -equivalence: M and N satisfy the same $L_{\infty\omega}^w$ sentences, where w is the universe of the index types.

Theorem 16 (Karp's theorem). M and N admit a potential isomorphism if and only if they are $L_{\infty\omega}^w$ -equivalent (satisfy the same $L_{\infty\omega}$ sentences with index types in universe w).

Proof. Forward: a potential isomorphism witnesses agreement on all $L_{\infty\omega}$ sentences by induction on formula complexity. Backward: BFEquiv at all ordinals gives a potential isomorphism via the game-tree family. \square

Chapter 4

Model Existence

Model existence and completeness for $\mathcal{L}_{\omega_1\omega}$, following Keisler [Kei71].

Theorem 17 (Model existence). *If a countable set S of $\mathcal{L}_{\omega_1\omega}$ sentences in a countable language belongs to a consistency property with equality axioms, then S has a countable model.*

Proof. Henkin construction: extend the language with constants, extend S to a maximal consistent set, build a term model, verify via truth lemma. \square

Theorem 18 (Karp completeness). *A sentence in a countable language that belongs to some consistency property with equality axioms has a countable model.*

Proof. Specialize the model existence theorem to the singleton theory $\{\varphi\}$ and extract the sentence from the resulting model. \square

4.1 Omitting Types

Definition 19 (Omits type). A structure M omits a type p (a set of formulas in one free variable) if no element of M realizes all formulas in p simultaneously.

Theorem 20 (Omitting types). *Given a countable consistent theory T in a countable language and a countable collection of non-isolated types, there exists a countable model of T that omits all the given types.*

Proof. Build a term model via the model existence theorem, then derive a contradiction for any element that would realize all formulas in an omitted type. \square

Chapter 5

Löwenheim–Skolem

The downward Löwenheim–Skolem theorem for $\mathcal{L}_{\omega_1\omega}$.

Definition 21 (Naming function with constants). The naming function for the extended language $L[[M]]$: each element $m \in M$ is named by the constant symbol $\text{con}(m)$.

Theorem 22 (Downward Löwenheim–Skolem). *Given a countable model M of an $\mathcal{L}_{\omega_1\omega}$ sentence φ in a countable language, one can construct a countable model in the same universe as M via language expansion $L[[M]]$ and the model existence theorem.*

Proof. The key step is the language expansion: extend L with a constant for each element of M to form $L[[M]]$, which remains countable. A naming function witnessing the constants lets us build a consistency property in $L[[M]]$; the model existence theorem then produces a fresh countable term model N , which we restrict back to L . \square

Theorem 23 (Downward Löwenheim–Skolem for theories). *Given a countable model M of a countable $\mathcal{L}_{\omega_1\omega}$ theory T in a countable language, one can reconstruct a countable model in the same universe as M via language expansion $L[[M]]$ and the model existence theorem.*

Proof. Same language-expansion strategy as the sentence version: lift the entire theory to $L[[M]]$, use the naming function to build a consistency property witnessing all sentences, apply model existence for the countable term model, then restrict the $L[[M]]$ -structure back to L . \square

Chapter 6

Admissible Fragments

Compactness and completeness for countable admissible fragments of $\mathcal{L}_{\omega_1\omega}$, following Barwise [Bar75]. Background on the Nadel bound for admissible fragments can be found in Nadel [Nad74].

Definition 24 (Admissible fragment). An admissible fragment of $\mathcal{L}_{\omega_1\omega}$: a collection of sentences closed under subformulas, negation, quantification, and countable conjunction/disjunction, with a compactness axiom for finite subsets.

Theorem 25 (Barwise compactness). *If every A -finite subset of a theory $T \subseteq A$ has a model, then T has a model.*

Proof. Direct application of the compactness property of the admissible fragment to the finite-satisfiability hypothesis. \square

Theorem 26 (Barwise completeness II). *A countable theory in an admissible fragment that belongs to a consistency property has a countable model.*

Proof. Extract the consistency property witness and apply the model existence theorem to obtain a countable term model. \square

Chapter 7

Hanf Numbers

Hanf numbers measure how large a model a sentence must have before it has arbitrarily large models.

Definition 27 (Arbitrarily large models). A sentence φ has arbitrarily large models if for every cardinal κ there is a model of φ of cardinality $\geq \kappa$.

Definition 28 (Hanf bound). κ is a Hanf bound for φ if having a model of cardinality $\geq \kappa$ implies having arbitrarily large models.

Definition 29 (Hanf number). The Hanf number of a sentence φ : the least cardinal that is a Hanf bound.

Theorem 30 (Hanf number existence). *Every $\mathcal{L}_{\omega_1\omega}$ sentence has a Hanf bound (and therefore a Hanf number).*

Proof. Case split: if φ has arbitrarily large models then any cardinal is a bound; otherwise there is a cutoff cardinal above which no models exist, so the premise is vacuously false. \square

7.1 Morley–Hanf Bound

Status: conditional. This result depends on `MorleyHanfTransfer`, an explicit hypothesis packaging Erdős–Rado partition calculus and Ehrenfeucht–Mostowski functor machinery not yet formalized in Lean or Mathlib.

Theorem 31 (Morley-Hanf bound). *Conditional on the Morley-Hanf transfer hypothesis, \beth_{ω_1} is a Hanf bound for every $\mathcal{L}_{\omega_1\omega}$ sentence.*

Proof. Given a model of size $\geq \beth_{\omega_1}$, the transfer hypothesis directly yields arbitrarily large models. \square

Chapter 8

Counting Models

Counting the number of models of a Scott sentence up to isomorphism. The theorem `morley_counting_dichotomy` (a counting-models dichotomy in the style of Morley) is formalized in Lean but is not included as a blueprint node; it serves as a schematic boundary result.

Status: schematic. The dichotomy statement quantifies over cardinals and does not correspond to a single blueprint-trackable proof obligation.

Theorem 32 (Stabilization bound determines isomorphism). *If M stabilizes completely at $\alpha < \omega_1$ and $\sim_\alpha(\bar{a}, \bar{b})$ holds, then $M \cong N$.*

Proof. Complete stabilization at α lets us upgrade \sim_α to \sim_γ for all $\gamma < \omega_1$, then apply Karp's theorem. \square

Theorem 33 (Bounded Scott height determines isomorphism). *If all countable models of φ have Scott height $\leq \alpha < \omega_1$, then isomorphism is equivalent to \sim_α .*

Proof. The forward direction follows from \sim being an invariant of isomorphism. For the reverse, the Scott height bound gives complete stabilization at α for both models, so we apply the stabilization-bound theorem. \square

Chapter 9

Proof System for Admissible Fragments

A proof system for admissible fragments of $\mathcal{L}_{\omega_1\omega}$, following Barwise [Bar75].

Definition 34 (Derivability in admissible fragments). A sentence φ is derivable from theory T in admissible fragment A . The proof system includes structural, propositional, infinitary, quantifier (omega-rule), equality, and classical rules.

Definition 35 (A -consistency). A theory T is A -consistent if \perp is not derivable from T in fragment A .

Theorem 36 (Soundness of the proof system). *If φ is derivable from T in fragment A , then φ is true in any model of T equipped with a naming function.*

Proof. By induction on the derivation. The quantifier cases use the semantic roundtrip for `openBounds` and the naming function. \square

Definition 37 (Full Barwise fragment). A full Barwise fragment: an admissible fragment containing all $\mathcal{L}_{\omega_1\omega}$ sentences, equipped with the chain closure property for consistency.

Theorem 38 (Consistency property from a full Barwise fragment). *The family of A -consistent subsets of a full Barwise fragment forms a `ConsistencyPropertyEq`.*

Proof. Each consistency property axiom (C0–C7) is verified by assuming the extension is inconsistent and deriving a contradiction via the proof system rules. \square

Theorem 39 (Barwise completeness II (syntactic, full fragment)). *A countable A -consistent theory in a full Barwise fragment of a countable language has a countable model.*

Proof. Construct a `ConsistencyPropertyEq` from the full Barwise fragment, then apply the model existence theorem. \square

Chapter 10

Descriptive Set Theory

Borel complexity of satisfaction, back-and-forth equivalence, and isomorphism on the coding space for countable structures.

Definition 40 (Structure space). The coding space for countable L -structures on \mathbb{N} : for each relation query, whether the relation holds on that tuple.

Theorem 41 (Satisfaction of $\mathcal{L}_{\omega_1\omega}$ sentences is Borel). *For a countable relational language, the set of codes in the structure space satisfying a given $\mathcal{L}_{\omega_1\omega}$ sentence is measurable (Borel).*

Proof. By structural induction on the sentence, using that atomic formulas are clopen and the Borel σ -algebra is closed under countable unions and intersections. \square

Theorem 42 (BFEquiv classes are Borel). *The set of code pairs where \sim_α holds is measurable for $\alpha < \omega_1$.*

Proof. By transfinite induction on α , matching the definition of \sim : the zero case uses countable intersection over atomic indices, the successor case uses the induction hypothesis with countable quantifier alternation, and the limit case uses a countable intersection since $\alpha < \omega_1$. \square

Theorem 43 (Isomorphism is Borel under bounded Scott height). *If all countable models of φ have Scott height $\leq \alpha < \omega_1$, then the set of isomorphic pairs within $\text{Mod}(\varphi) \times \text{Mod}(\varphi)$ is measurable.*

Proof. The isomorphic-pair set restricted to $\text{Mod}(\varphi)$ equals the \sim_α set restricted to $\text{Mod}(\varphi)$, by the bounded Scott height theorem. Both components are measurable. \square

10.1 Standard Borel Structure

The structure space carries the product topology, which makes it a Polish space when the language has countably many relation symbols.

Theorem 44 (Structure space is Polish). *For a countable relational language, the structure space is a Polish space.*

Proof. The structure space is a countable product of copies of $\{0, 1\}$, which is compact and metrizable, hence Polish. \square

The model class of any $\mathcal{L}_{\omega_1\omega}$ sentence inherits a standard Borel structure.

Theorem 45 (Model class is clopenable). *For any $\mathcal{L}_{\omega_1\omega}$ sentence φ , the set $\text{Mod}(\varphi)$ is clopenable in the structure space.*

Proof. $\text{Mod}(\varphi)$ is measurable (by satisfaction is Borel) and the structure space is Polish + Borel, so any measurable set is clopenable. \square

Theorem 46 (Model class is standard Borel). *For any $\mathcal{L}_{\omega_1\omega}$ sentence φ , the subtype $\text{Mod}(\varphi)$ is a standard Borel space.*

Proof. A measurable subset of a standard Borel space is standard Borel. \square

10.2 Counting Models

The Silver–Burgess dichotomy, taken as a hypothesis, yields a counting theorem for coded \mathbb{N} -models with bounded Scott height.

Definition 47 (Silver–Burgess dichotomy (hypothesis)). *On any standard Borel space, a Borel equivalence relation has either at most countably many equivalence classes or exactly 2^{\aleph_0} .*

Definition 48 (Isomorphism setoid on coded models). *The equivalence relation on coded \mathbb{N} -models of φ where two codes are related iff their decoded structures are L -isomorphic.*

Theorem 49 (Conditional counting dichotomy). *If the Silver–Burgess dichotomy holds, then for any $\mathcal{L}_{\omega_1\omega}$ sentence whose \mathbb{N} -models all have Scott height $\leq \alpha < \omega_1$, the number of isomorphism classes among coded \mathbb{N} -models is either $\leq \aleph_0$ or exactly 2^{\aleph_0} .*

Proof. The model class is standard Borel and the isomorphism relation is Borel (by bounded Scott height), so Silver–Burgess applies directly. \square

Theorem 50 (Isomorphism is Borel on finite carriers). *For structures on $\text{Fin } n$, the isomorphism relation is the orbit of $\text{Sym}(\text{Fin } n)$, a finite union of graphs of continuous maps, hence Borel.*

Proof. \square

Theorem 51 (Counting dichotomy for all countable models). *If the Silver–Burgess dichotomy holds, then for any $\mathcal{L}_{\omega_1\omega}$ sentence whose countable models all have Scott height $\leq \alpha < \omega_1$, the total number of isomorphism classes of countable models is either $\leq \aleph_0$ or exactly 2^{\aleph_0} .*

Proof. Combine the \mathbb{N} -coded result (via BF-equivalence Borelness) with finite-carrier orbit arguments for each $\text{Fin } n$ tier. \square

Theorem 52 (Morley’s counting theorem). *For any $\mathcal{L}_{\omega_1\omega}$ sentence φ , the number of isomorphism classes of countable models is either $\leq \aleph_1$ or exactly 2^{\aleph_0} .*

Proof. Stratify by Scott height. For each $\alpha < \omega_1$, BF-equivalence at level α is Borel with $\leq \aleph_0$ or 2^{\aleph_0} classes. Iso classes with height $\leq \alpha$ inject into BF classes. Union over ω_1 strata gives $\leq \aleph_1$. \square

The combined theorem extends the coded \mathbb{N} -model result to all countable models by adding finite-carrier tiers (isomorphism on $\text{Fin } n$ structures is Borel via the orbit of $\text{Sym}(n)$). Bridge theorems (`codeModel`, `iso_of_codeModel_eq`, `codeModel_surjective`) establish that `AllCodedIsoClasses` faithfully represents all isomorphism classes of countable models. The Silver–Burgess dichotomy (`silverBurgessDichotomy`) is proved modulo one sorry in the Borel-to-closed reduction (`silver_core_polish`), which requires the Gandy–Harrington topology or equivalent descriptive-set-theoretic infrastructure not yet available in Mathlib. Once this is filled, `morley_counting` becomes fully unconditional.

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